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Ruelle operator with nonexpansive IFS on the line ☆

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Abstract

Ruelle operator defined by weakly contractive iterated function systems (IFS) satisfying the open set condition was discussed in the paper [K.S. Lau, Y.L. Ye, Ruelle operator with nonexpansive IFS, Studia Math. 148 (2001) 143–169]. There, one of our theorems gave a sufficient condition for the possession of the Perron–Frobenius property. In this paper we consider Ruelle operator defined by nonexpansive IFS on the line instead of by weakly contractive one. And we prove, under the same condition, that the newly defined Ruelle operator has the Perron–Frobenius property. It extends the Ruelle–Perron–Frobenius theorem partially to the nonexpansive IFS.

Keywords: Ruelle operator; Nonexpansive map; Weakly contractive map; Perron–Frobenius property

1. Introduction

Let $\{w_j\}_{j=1}^m$ be a weakly contractive IFS on a compact subset $X \subseteq \mathbb{R}^d$. With each w_j we associate a positive Dini function p_j as weight function; the triple $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is called a *weakly contractive Dini system*. Define the *Ruelle operator* on $C(X)$ as

$$T(f)(x) = \sum_{j=1}^{m} p_j\big(w_j(x)\big) f\big(w_j(x)\big), \qquad f \in C(X). \tag{1.1}$$

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Let ϱ denote the spectral radius of T. The Ruelle operator and the associated Perron–Frobenius property have been studied extensively in wavelets, dynamical systems and IFS (see, e.g., [1,2,4–7,9–12]). The importance and the necessity of studying the Ruelle operators can be found in [8], and the notations and concepts are adopted from there.

In this paper we will always consider the IFS $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ where X is a nonempty compact set with $\overline{X^\circ} = X$, each w_j is a weakly contractive or nonexpansive self-map on X, and each p_j is a positive Dini continuous weight function. For the weakly contractive case, Hata [3] showed that there exists a unique compact subset K such that $K = \bigcup_{j=1}^m w_j(K)$. For the nonexpansive case we can take the smallest invariant set K (see Proposition 2.1 for the additional assumption). Hence, in both cases, we can set up the Ruelle operator as in (1.1) on $C(K)$. We say that the Ruelle operator T on $C(K)$ has the *PF-property* (PF stands for Perron–Frobenius) if there exist a unique $0 < h \in C(K)$ and a unique probability $\mu \in M(K)$ such that

$$Th = \varrho h, \qquad T^*\mu = \varrho\mu, \qquad \langle \mu, h\rangle = 1,$$

and for every $f \in C(K)$, $\varrho^{-n}T^n f$ converges to $\langle \mu, f\rangle h$ in the supremum norm.

At the end of the paper [8], we asked whether the weakly contraction can be replaced by the nonexpansion in [8, Theorem 5.1]. In this paper we answer affirmatively the question partially. Precisely, we show the following result.

Theorem. *Let w_j $(1 \leqslant j \leqslant m)$ be nonexpansive self-maps on the closed interval* $[0, 1]$ *satisfying the following assumptions*:

(i) $w_i(0,1) \subseteq (0,1)$ *and* $w_i(0,1) \cap w_j(0,1) = \emptyset$ $\forall i \neq j$;
(ii) $w_j'(x)$*s are Dini continuous and satisfy the condition*

$$0 < \inf_{x,j} \left|w_j'(x)\right| \leqslant \sup_{x,j} \left|w_j'(x)\right| \leqslant 1.$$

Suppose that for some $0 \leqslant l \leqslant m$ and for all $1 \leqslant j \leqslant l$, w_j are contractive and

$$\max_{\substack{l+1 \leqslant j \leqslant m \\ x \in X}} p_j(w_j x) < \varrho.$$

Then T has the PF-property.

We remark that the above theorem was considered by Öberg [10] for w_js weakly contractive and p_js Hölder continuous. In general it is difficult to determine the spectral radius ϱ in the above theorem. We give a simple checkable sufficient condition in Corollary 3.2.

We organize the paper as follows. In Section 2, we will present some elementary facts about the Ruelle operator. In Section 3 we prove the above theorem.

2. Preliminaries

We say that a self-map $w: X \to X$ is *nonexpansive* if $|w(x) - w(y)| \leqslant |x - y|$, and *weakly contractive* if

$$\alpha_w(t) := \sup_{|x-y| \leqslant t} \left|w(x) - w(y)\right| < t \quad \forall t > 0.$$

It is clear that contractivity implies weak contractivity, which also implies nonexpansiveness. A simple nontrivial example of weakly contractive map is $w(x) = x/(1+x)$ on [0, 1]. We call a function $p : X \to \mathbb{R}^+$ Dini continuous if $\int_0^1 \alpha_p(t)t^{-1}\,dt < \infty$.

Throughout this section, we let $\{w_j\}_{j=1}^m$ be a nonexpansive IFS on X. Let $\{p_j\}_{j=1}^m$ be a set of Dini continuous positive weight functions associated with the IFS $\{w_j\}_{j=1}^m$. We call the triple $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a *nonexpansive Dini system* if w_js are nonexpansive. Similarly we can define the corresponding terminology for the weakly contractive IFS. Define the *Ruelle operator* $T : C(X) \to C(X)$ by

$$Tf(x) = \sum_{j=1}^m p_j\big(w_j(x)\big) f\big(w_j(x)\big).$$

For any multi-index $J = (j_1 j_2 \cdots j_n)$, we let $|J| = n$, and let $w_J(x) = w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x)$. Let

$$p_{w_J}(x) = p_{j_1}\big(w_{j_1} \circ w_{j_2} \circ \cdots \circ w_{j_n}(x)\big) \cdots p_{j_n}\big(w_{j_n}(x)\big).$$

We can confirm inductively that

$$T^n f(x) = \sum_{|J|=n} p_{w_J}(x) f\big(w_J(x)\big).$$

Let $\varrho = \varrho(T)$ be the spectral radius of T. Then

$$\varrho = \lim_{n\to\infty} \|T^n\|_{C(X)}^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\max_{x\in X} T^n 1(x)\Big)^{\frac{1}{n}}.$$

In [3] Hata has studied the invariant sets of the weakly contractive IFS on X. He showed the existence of a unique nonempty compact $K \subseteq X$ invariant under w_js, i.e., $K = \bigcup_{j=1}^m w_j(K)$, and $K = \bigcap_{n=1}^\infty \bigcup_{|J|=n} w_J(K)$. Moreover, $\lim_{|J|\to\infty} |w_J(K)| = 0$. For the more general IFS, the invariant set may not be unique. However, similar to [8, Proposition 2.1], we have

Proposition 2.1. *Let $\{w_j\}_{j=1}^m$ be a nonexpansive IFS on X. Suppose that there exists at least a multi-index $J_0 \in \bigcup_{n=1}^\infty \{1, 2, \ldots, m\}^n$ such that the map w_{J_0} is weakly contractive on X. Then there exists a unique smallest nonempty compact set K such that*

$$K = \bigcup_{j=1}^m w_j(K).$$

Moreover, for any $x \in K$, the closure of $\{w_J(x)\colon |J| = n,\ n \in \mathbb{N}\}$ is K.

Proof. The proof is modified from [8, Proposition 2.1]. We include the details here for completeness. Let $\mathcal{F} = \{F\colon \bigcup_{j=1}^m w_j(F) \subseteq F\}$. By using the standard Zorn's lemma argument, there exists a minimal nonempty compact subset K such that $K = \bigcup_{j=1}^m w_j(K)$. To show that such K is unique, we denote $\mathcal{J}_n = (\overbrace{J_0 J_0 \cdots J_0}^{n})$, then $\lim_{n\to\infty} |w_{\mathcal{J}_n}(X)| = 0$ because the map w_{J_0} is weakly contractive. Let K' be another minimal nonempty compact invariant set, and let $x \in K$ and $y \in K'$. Then

$$\lim_{n\to\infty} w_{\mathcal{J}_n}(x) = \lim_{n\to\infty} w_{\mathcal{J}_n}(y) \in K \cap K'.$$

Hence $K \cap K' \neq \emptyset$ and $w_j(K \cap K') \subseteq K \cap K'$. The minimality implies that $K = K'$.

The last statement follows from the fact that K is the smallest invariant subset. □

Hence under the assumption of Proposition 2.1, the set K is uniquely defined. We remark that, in the both cases, we can consider the Ruelle operator $T|_{C(K)}$ restricted on $C(K)$, i.e., $T : C(K) \to C(K)$ defined by

$$Tf(x) = \sum_{j=1}^{m} p_j\big(w_j(x)\big) f\big(w_j(x)\big).$$

Proposition 2.2. *Suppose that $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ is a nonexpansive Dini system, and let $R_J = \sup_{x,y\in X,\ x\neq y}(|w_J(x) - w_J(y)|/|x-y|)$. If there exists an integer k such that*

$$\max_{x\in X} \sum_{|J|=k} p_{w_J}(x) R_J < \varrho^k, \tag{2.1}$$

then T has the PF-property.

Proof. By (2.1), there exists some multi-index J_0 with $|J_0| = k$ such that $R_{J_0} < 1$. This implies that the map w_{J_0} is contractive on X. By Proposition 2.1, there exists a unique smallest nonempty compact set K such that $K = \bigcup_{j=1}^m w_j(K)$.

By using (2.1), in a way similar to that in [8, Theorem 4.4], we can find constants $B \geqslant A > 0$ such that

$$A \leqslant \varrho^{-n} T^n 1(x) \leqslant B \quad \forall x \in X.$$

This implies that

$$\varrho = \lim_{n\to\infty} \big\|T^n\big\|_{C(K)}^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\max_{x\in K} T^n 1(x)\Big)^{\frac{1}{n}}.$$

Hence, by [8, Theorem 4.4], it follows that T has the PF-property. □

3. Continuously differentiable IFS on the line

Throughout this section, we always let $\{w_j\}_{j=1}^m$ be a nonexpansive and continuously differentiable IFS defined on a closed interval $X (\subset \mathbb{R})$. Without loss of generality, we may assume that $X = [0, 1]$. We assume that $|w_j'(x)|$s are Dini continuous and satisfy the condition

$$0 < \inf_{x,j}\big|w_j'(x)\big| \leqslant \sup_{x,j}\big|w_j'(x)\big| \leqslant 1.$$

We also assume that the IFS $\{w_j\}_{j=1}^m$ has nonoverlapping in the sense that

$$w_i\big(X^\circ\big) \cap w_j\big(X^\circ\big) = \emptyset \quad \forall i \neq j.$$

For any multi-index J, let $R_J = \sup_{x\in X} |w_J'(x)|$. Then

$$R_J = \sup_{x\neq y} \frac{|w_J(x) - w_J(y)|}{|x-y|}.$$

We remark that, in the proof of Theorem 5.1 of [8], we need to use the weak contractivity of w_js ($d_n := \max_{|J|=n} |w_J(X)| \to 0$). In the following Theorem 3.1, we will show that it can be replaced by the nonexpansion.

Theorem 3.1. *Let the IFS $\{w_j\}_{j=1}^m$ be as above, and let $\{p_j\}_{j=1}^m$ be a set of Dini continuous positive weight functions associated with the IFS. If for some $0 \leqslant l \leqslant m$ and for all $1 \leqslant j \leqslant l$, w_j are contractive and*

$$\max_{\substack{l+1 \leqslant j \leqslant m \\ x \in X}} p_j(w_j x) < \varrho, \tag{3.1}$$

then T has the PF-property.

Proof. We divide the proof in two cases: (i) $l = 0$ (i.e., none of the maps are contractive) and (ii) $l \geqslant 1$.

(i) $l = 0$: Let $\delta(t) = \max_{1 \leqslant j \leqslant m} \alpha_{w'_j}(t)$. Since $|w'_j(x)|$ is Dini continuous and positive on the compact set X, there exists $c_1 > 0$ such that

$$\frac{|w'_j(x)|}{|w'_j(y)|} \leqslant 1 + c_1 \delta\bigl(|x-y|\bigr) \quad \forall x, y \in X.$$

For any multi-index I, we let $X_I = w_I(X)$. Then for any $J = (j_1 j_2 \cdots j_n)$,

$$\begin{aligned} \frac{|w'_J(w_I(x))|}{|w'_J(w_I(y))|} &= \prod_{k=1}^{n} \frac{|w'_{j_k}(w_{j_{k+1}} \circ w_{j_{k+2}} \circ \cdots \circ w_{j_n}(w_I x))|}{|w'_{j_k}(w_{j_{k+1}} \circ w_{j_{k+2}} \circ \cdots \circ w_{j_n}(w_I y))|} \\ &\leqslant \prod_{k=0}^{n-1} \bigl(1 + c_1 \delta\bigl(\bigl|w_{J|_k^n}(X_I)\bigr|\bigr)\bigr), \end{aligned}$$

where $J|_k^n = (j_{k+1} \cdots j_n)$.

By the assumptions on the IFS $\{w_j\}_{j=1}^m$, we have

$$\inf_{x \in X} \bigl|w'_J\bigl(w_I(x)\bigr)\bigr| \leqslant \bigl|w_J(X)\bigr|.$$

(We use $X \subset \mathbb{R}$ here.) Thus we have

$$\sup_{x \in X} \bigl|w'_J\bigl(w_I(x)\bigr)\bigr| \leqslant \bigl|w_J(X)\bigr| \cdot \prod_{k=0}^{n-1} \bigl(1 + c_1 \delta\bigl(\bigl|w_{J|_k^n}(X_I)\bigr|\bigr)\bigr). \tag{3.2}$$

Let

$$c = \max_{\substack{l+1 \leqslant j \leqslant m \\ x \in X}} p_j(w_j x).$$

Choose $\theta, \varepsilon > 0$ such that $c\varrho^{-1} < \theta < 1$ and $\theta e^{\varepsilon} < 1$. By the Dini continuity of $|w'_j(\cdot)|$s, we have

$$\lim_{t \to 0^+} \delta(t) = \delta(0) = 0.$$

Then there exists $t_0 > 0$ such that

$$\delta(t) < {c_1}^{-1} \varepsilon \quad \forall t \leqslant t_0. \tag{3.3}$$

For any multi-index I satisfying the inequality $|w_I(X)| < t_0$, we conclude from (3.3) that

$$\prod_{k=0}^{n-1} \bigl(1 + c_1 \delta\bigl(\bigl|w_{J|_k^n}(X_I)\bigr|\bigr)\bigr) \leqslant e^{n\varepsilon} = e^{\varepsilon |J|}.$$

From this, together with (3.2), we deduce that

$$\sup_{x\in X}\left|w_J'\big(w_I(x)\big)\right| \leqslant e^{\varepsilon|J|}\cdot\left|w_J(X)\right|. \tag{3.4}$$

Let

$$\Omega(n)=\left\{J\colon\ |J|=n,\ \left|w_J(X)\right|\leqslant\theta^n\right\},\qquad \Omega'(n)=\left\{J\colon\ |J|=n,\ \left|w_J(X)\right|>\theta^n\right\}.$$

Since the IFS $\{w_j\}_{j=1}^m$ has nonoverlapping, we have

$$w_I(X^\circ)\cap w_J(X^\circ)=\emptyset\quad \forall I\neq J \text{ with } |I|=|J|.$$

From this, together with the fact that $|w_J(X^\circ)|=|w_J(X)|$ $\forall J$, we deduce that $\#\Omega'(n)<\theta^{-n}$. (We use $X\subset\mathbb{R}$ here.)

For any integers n, ℓ and $0\leqslant k\leqslant\ell$, we denote

$$A(\ell,n,k)=\left\{J=(J_1J_2\cdots J_\ell)\colon\ |J_i|=n\ \forall i \text{ and } \#\left\{i\colon\ J_i\in\Omega'(n)\right\}=k\right\}.$$

It is easy to confirm that

$$\left\{J\colon\ |J|=\ell n\right\}=\bigcup_{k=0}^{\ell}A(\ell,n,k).$$

By $\#\Omega'(n)<\theta^{-n}$, we have

$$\sum_{J\in\Omega'(n)}p_{w_J}(x)\leqslant\left(c\theta^{-1}\right)^n. \tag{3.5}$$

Choose a constant $0<a<1$ such that $c\theta^{-1}<a\varrho<\varrho$. We can choose n_0 large enough such that $\theta^n<t_0$ and

$$\left\|T^n\right\|\geqslant(a\varrho)^n\quad \forall n\geqslant n_0.$$

For $0\leqslant k<\ell$ and $J=(J_1J_2\cdots J_\ell)\in A(\ell,n,k)$, we let p be the smallest integer such that

$$\left|w_{J_{\ell-p}}(X)\right|\leqslant\theta^n.$$

(We remark $p\leqslant k$ here.) Then by (3.4), we have

$$\begin{aligned} R_J&\leqslant\sup_{x\in X}\prod_{j=p+1}^{\ell-1}\left|w'_{J_{\ell-j}}\big(w_{J_{\ell-j+1}}\circ\cdots\circ w_{J_{\ell-p}}(x)\big)\right| \\ &\leqslant\left(e^{n\varepsilon}\right)^{\ell-k-1}\left(\theta^n\right)^{\ell-k-1}\leqslant\left(\theta e^{\varepsilon}\right)^{\ell n-kn-n}. \end{aligned}$$

At the same time, we have

$$\begin{aligned} \sum_{J\in A(\ell,n,k)}p_{w_J}(x)&\leqslant C_\ell^k\cdot\left(\max_{x\in X}\sum_{I\in\Omega'(n)}p_{w_I}(x)\right)^k\cdot\left(\max_{x\in X}\sum_{J\in\Omega(n)}p_{w_J}(x)\right)^{\ell-k} \\ &\leqslant C_\ell^k\left(c\theta^{-1}\right)^{kn}\left\|T^n\right\|^{\ell-k}\quad\big(\text{by (3.5)}\big) \\ &\leqslant C_\ell^k\left(c\theta^{-1}\right)^{kn}\frac{\|T^n\|^k}{(a\varrho)^{kn}}\left\|T^n\right\|^{\ell-k}=C_\ell^k\left(\frac{c}{a\varrho\theta}\right)^{kn}\left\|T^n\right\|^{\ell}. \end{aligned}$$

Let $\tau = c(a\varrho\theta)^{-1}$. Then $0 < \tau < 1$ and

$$\sum_{J\in A(\ell,n,k)} p_{w_J}(x)R_J \leqslant C_\ell^k \tau^{kn} \left\| T^{\ell n} \right\| \cdot \left(\theta e^{\varepsilon}\right)^{\ell n-kn-n}.$$

This implies that

$$\limsup_{n\to\infty}\left(\max_{x\in X}\sum_{J\in A(\ell,n,k)} p_{w_J}(x)R_J\right)^{\frac{1}{n}} \leqslant \varrho^\ell \tau^k \left(\theta e^{\varepsilon}\right)^{\ell-k-1}. \tag{3.6}$$

By (3.5), we have

$$\sum_{J\in A(\ell,n,\ell)} p_{w_J}(x) \leqslant \left(\max_{x\in X}\sum_{J\in\Omega'(n)} p_{w_J}(x)\right)^\ell \leqslant (c\theta^{-1})^{\ell n}.$$

Then

$$\limsup_{n\to\infty}\left(\max_{x\in X}\sum_{J\in A(\ell,n,\ell)} p_{w_J}(x)R_J\right)^{\frac{1}{n}} \leqslant \left(c\theta^{-1}\right)^\ell.$$

This, together with (3.6), implies that

$$\begin{aligned}
&\limsup_{n\to\infty}\left(\max_{x\in X}\sum_{|J|=\ell n} p_{w_J}(x)R_J\right)^{\frac{1}{n}}\\
&\quad= \limsup_{n\to\infty}\left(\max_{x\in X}\sum_{k=0}^{\ell}\sum_{J\in A(\ell,n,k)} p_{w_J}(x)R_J\right)^{\frac{1}{n}}\\
&\quad\leqslant \left(c\theta^{-1}\right)^\ell + \sum_{k=0}^{\ell-1}\varrho^\ell\tau^k\left(\theta e^{\varepsilon}\right)^{\ell-k-1} := b_\ell.
\end{aligned}$$

By a direct calculation, we can deduce that

$$\limsup_{\ell\to\infty}(b_\ell)^{\frac{1}{\ell}} \leqslant \max\left\{c\theta^{-1}, \theta e^{\varepsilon}\varrho, \tau\varrho\right\} < \varrho.$$

This implies that

$$\limsup_{\ell\to\infty}\limsup_{n\to\infty}\left(\max_{x\in X}\sum_{|J|=\ell n} p_{w_J}(x)R_J\right)^{\frac{1}{\ell n}} < \varrho.$$

Hence, there exist integers $\ell, n \in \mathbb{N}$ such that

$$\left(\sum_{|J|=\ell n} p_{w_J}(x)R_J\right)^{\frac{1}{\ell n}} < \varrho.$$

Proposition 2.2 applies and T has the PF-property.

(ii) $l \geqslant 1$: The theorem is proved in [2] if all the maps are contractive, hence we assume that $1 \leqslant l \leqslant m-1$. Let c be defined as above and let

$$b = \max_{\substack{1\leqslant j\leqslant l\\ x\in X}} p_j(w_j x), \qquad R = \max_{\substack{1\leqslant j\leqslant l\\ x\in X}} \left|w_j'(x)\right| < 1.$$

If $b \leqslant c$, then

$$\max_{\substack{1 \leqslant j \leqslant m \\ x \in X}} p_j(w_j x) = \max_{\substack{l+1 \leqslant j \leqslant m \\ x \in X}} p_j(w_j x) < \varrho,$$

and the proof of (i) applies. Hence we only consider $b > c$. We choose $0 < a < 1$, λ and θ such that

$$c\varrho^{-1} < \theta < 1 \quad \text{and} \quad \theta^{-1}\left(bc^{-1}\right)^{\log\theta/\log R} < \lambda < a\varrho c^{-1}.$$

Let $\Omega(n)$, $\Omega'(n)$ and $A(\ell, n, k)$ be defined as above. For any $J \in \Omega'(n)$, denote

$$k_J = \#\left\{j_i\colon\ J = (j_1 j_2 \cdots j_i \cdots j_n) \in \Omega'(n),\ 1 \leqslant j_i \leqslant l\right\}.$$

Then $p_{w_J}(x) \leqslant c^{n-k_J} b^{k_J}$ and $\#\Omega'(n) \leqslant \theta^{-n}$ as in (i), and

$$\sum_{J\in\Omega'(n)} p_{w_J}(x) \leqslant \theta^{-n} \max_{J\in\Omega'(n)} \left(c^{n-k_J} b^{k_J}\right) = \left(c\theta^{-1}\right)^n \max_{J\in\Omega'(n)} \left(bc^{-1}\right)^{k_J}. \tag{3.7}$$

Let $k_n = \max\{k_J\colon\ J \in \Omega'(n)\}$. Since $\theta^n < |w_J(X)| \leqslant R^{k_J}$ for any $J \in \Omega'(n)$, this implies that $\theta^n \leqslant R^{k_n}$ and

$$\left(\max_{J\in\Omega'(n)} \left(bc^{-1}\right)^{k_J}\right)^{\frac{1}{n}} \leqslant \left(bc^{-1}\right)^{\frac{k_n}{n}} \leqslant \left(bc^{-1}\right)^{\frac{\log\theta}{\log R}} < \theta\lambda.$$

This, together with (3.7), implies that

$$\limsup_{n\to\infty} \left(\max_{x\in X} \sum_{J\in\Omega'(n)} p_{w_J}(x)\right)^{\frac{1}{n}} < \left(c\theta^{-1}\right)(\theta\lambda) = c\lambda.$$

Choose an integer n_0 such that for any $n \geqslant n_0$, we have $\theta^n < t_0$, and

$$\sum_{J\in\Omega'(n)} p_{w_J}(x) < (c\lambda)^n, \tag{3.8}$$

$$\left\|T^n\right\| \geqslant (a\varrho)^n. \tag{3.9}$$

Similar to (3.6), we can deduce from (3.8) and (3.9) that for any $0 \leqslant k < \ell$,

$$\limsup_{n\to\infty} \left(\max_{x\in X} \sum_{J\in A(\ell,n,k)} p_{w_J}(x) R_J\right)^{\frac{1}{n}} \leqslant \varrho^\ell \left(\frac{c\lambda}{a\varrho}\right)^k \left(\theta e^\varepsilon\right)^{\ell-k-1}.$$

By (3.8), we have

$$\sum_{J\in A(\ell,n,\ell)} p_{w_J}(x) \leqslant \left(\max_{x\in X} \sum_{J\in\Omega'(n)} p_{w_J}(x)\right)^\ell \leqslant (c\lambda)^{\ell n}.$$

Hence

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\max_{x\in X} \sum_{|J|=\ell n} p_{w_J}(x) R_J\right)^{\frac{1}{n}} \\
&\quad = \limsup_{n\to\infty} \left(\max_{x\in X} \sum_{k=0}^{\ell} \sum_{J\in A(\ell,n,k)} p_{w_J}(x) R_J\right)^{\frac{1}{n}} \\
&\quad \leqslant (c\lambda)^\ell + \sum_{k=0}^{\ell-1} \varrho^\ell \left(\frac{c\lambda}{a\varrho}\right)^k \left(\theta e^\varepsilon\right)^{\ell-k-1} := d_\ell.
\end{aligned}$$

By a direct calculation, we deduce that

$$\limsup_{\ell\to\infty}\limsup_{n\to\infty}\left(\max_{x\in X}\sum_{|J|=\ell n} p_{w_J}(x)R_J\right)^{\frac{1}{\ell n}} \leqslant \limsup_{\ell\to\infty}(d_\ell)^{\frac{1}{\ell}} < \varrho.$$

Again by Proposition 2.2, T has the PF-property. This completes the proof. □

It is obvious that if $\{w_j\}_{j=1}^m$ are contractive maps, then the condition in the theorem is trivially satisfied. In general, it is difficult to determine the spectral radius ϱ of T. A simple estimation on the lower bound of ϱ is

$$\min_{x\in X}\left(\sum_{j=1}^m p_j(w_j x)\right) \leqslant \varrho.$$

As a direct consequence, we have the following corollary.

Corollary 3.2. *Suppose that for some $0 \leqslant l \leqslant m$ and for all $1 \leqslant j \leqslant l$, w_j are contractive and*

$$\max_{\substack{l+1\leqslant j\leqslant m,\\ x\in X}} p_j(w_j x) < \min_{x\in X}\left(\sum_{j=1}^m p_j(w_j x)\right).$$

Then T has the PF-property.

We would like to point out that Theorem 3.1 in this paper is a generalization of [8, Theorem 5.1]. However, the following example indicates that this generalization is a nontrivial one.

Example 3.3. Let $X = [0, 2]$, and let

$$w_1(x) = \frac{x}{1+x} \quad \forall x \in X,$$

$$w_2(x) = \begin{cases} 1+x & \text{if } 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{4}{3} + \frac{x-\frac{1}{3}}{x+\frac{2}{3}} & \text{if } \frac{1}{3} < x \leqslant 2. \end{cases}$$

Then, $(X, \{w_j\}_{j=1}^2)$ is a continuously differentiable IFS which has nonoverlapping. Note that $w_1'(0) = w_2'(0) = 1$. It is easy to confirm that the map w_1 is weakly contractive, and that the map w_2 is nonexpansive. Moreover, we have

$$\lim_{n\to\infty}\left|w_2^n(2) - w_2^n(0)\right| = \frac{1}{3}.$$

This implies that the map w_2 is not weakly contractive.

Let p_1 be a positive Dini function on X with the inequalities $0 < p_1(\cdot) < \frac{1}{2}$. Let $a = \frac{1}{2} \cdot \min_{x\in X} p_1(x)$. Define $g : X \to [0, a]$ by

$$g(x) = \begin{cases} a - 1 + x & \text{if } 1 - a < x \leqslant 1, \\ a + 1 - x & \text{if } 1 < x < 1 + a, \\ 0 & \text{otherwise.} \end{cases}$$

Let p_2 be a Dini function on X such that

$$p_2 \circ w_2(x) = 1 - p_1 \circ w_1(x) + g(x) \quad \forall x \in X.$$

Let T be the Ruelle operator defined by the IFS $(X, \{w_j\}_{j=1}^2, \{p_j\}_{j=1}^2)$ as in (1.1). It is easy to confirm that for any $x \in X$,

$$0 < \min\bigl\{p_1 \circ w_1(x), p_2 \circ w_2(x)\bigr\} \leqslant \max\bigl\{p_1 \circ w_1(x), p_2 \circ w_2(x)\bigr\} < 1,$$

and that

$$\sum_{j=1}^{2} p_j(w_j x) = 1 + g(x) \geqslant 1.$$

Hence, Corollary 3.2 implies that T has the PF-property.

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